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Bayesian Estimation For Modulated Claim Hedging

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Abstract
The purpose of this paper is to establish a general super hedging formula under a pricing set Q. We will compute the price and the strategies for hedging an European claim and simulate that using different approaches including Dirichlet priors. We study Dirichlet processes centered around the distribution of continuous-time stochastic processes such as a continuous time Markov chain. We assume that the prior distribution of the unobserved Markov chain driving by the drift and volatility parameters of the geometric Brownian motion (GBM) is a Dirichlet process. We propose an estimation method based on Gibbs sampling.

I. Introduction

Models in which parameters move between a fixed number of regimes with switching controlled by an unobserved stochastic process, are very popular in a great variety of domains (Finance, Biology, Meteorology, Networks, etc.). This is notably due to the fact that this additional flexibility allows the model to account for random regime changes in the environment. In this paper we consider the estimation problem for a model described by a stochastic differential equation (SDE) with Markov regime-switching (MRS), i.e., with parameters controlled by a finite state continuous-time Markov chain (CTMC). Such a model was used, for example, in Deshpande and Ghosh (2008) to price options in a regime switching market. In such a setting, the parameter estimation problem poses a real challenge, mainly due to the fact that the paths of the CTMC are unobserved. A standard approach consists in using the celebrated EM algorithm (Dempster, Laird, Rubin, 1977) as proposed for example in Hamilton (1990). Elliott, Malcolm and Tsoi (2003) study this problem using a filtering approach.

In the first part of work, we consider $(\Omega,\mathsf{F},\mathsf{P})$ be a probability space endowed with a filtration $(\mathsf{F}_t)_{t\in\mathsf{T}}$ with $\mathsf{T}=[0,T]$ and T is the time Horizon. We consider a financial market which is composed of a risky asset with the price process $S=(S_t)_{t\in\mathsf{T}}$ and a risk free asset taken as a numéraire with the price process is equal to one at any time for simplicity. Let $\mathsf{M}(S)$ be the set of all P -absolutely continuous martingale measures of the process S and $\mathsf{M}^e(S)$ the subset of the equivalent ones. We suppose that $\mathsf{M}^e(S)\neq\varnothing$ and consider the pricing set Q as the collection of probabilities $\mathsf{Q}\in\mathsf{M}(S)$ that satisfy $\mathsf{Q}_t(X_T)\in[c_t,d_t]$ for all $t\in\mathsf{T}$ for some fixed processes c and d such that $-m_t(-X_T)<c_t\le d_t<m_t(X_T)$ with $m_t(X)=esssup_{\mathsf{Q}\in\mathsf{M}(S)}\mathsf{Q}_t(X)$ and X_T is the payoff of a fixed positive European claim.

Our goal in this project can be decomposed in two parts:

1. In the first part we will establish a general super hedging formula under the pricing set Q. We will state that for any positive Q -super martingale $(Y_t)_{t\in\mathsf{T}}$, there exists some $\mathsf{Q}\in\mathsf{Q}$, an R^3 -valued predictable process $(\alpha_t)_{t\in\mathsf{T}}$ and an increasing process $(C_t)_{t\in\mathsf{T}}$ such that for all $t\in\mathsf{T}$ we have:

$$Y_t=Y_0+\int_0^t\alpha_s.dV_s^{\mathsf{Q}}-C_t,$$

with

$$\mathsf{Q}'_t\left[\int_0^T\alpha_s.dV_s^{\mathsf{Q}}\right]\le\int_0^t\alpha_s.dV_s^{\mathsf{Q}},$$

for all $\mathsf{Q}'\in\mathsf{Q}$ with $V_t^{\mathsf{Q}}=\mathsf{Q}_t(V)$ and $V=(1,S_T,X_T)$. Our work will be based mainly on the optional decomposition theorem stated in [1].

2. In the second part we apply the results of part one to the following model: we suppose that the process S is solution of the stochastic differential equation:

$$dS_t=\mu S_t\,dt+\sqrt{Y_t}\,S_t\,dW_t,$$

and the process Y satisfies the stochastic differential equation:

$$dY_t=a(b-Y_t)dt+\sigma\sqrt{Y_t}\,dB_t,$$

where W, B are two correlated Brownian motions, μ , a and b are real parameters. Following notations of the first part we define $X_T = \max(S_T - K, 0)$ the payoff of an European call option with strike K , $c_t = -m_t(-X_T)$ and $d_t = \beta m_t(X_T)$ with $0 < \beta \leq 1$. We will compute the price and the strategies for hedging an European claim and simulate that using different approaches including Dirichlet priors.

In the second part of this paper, we propose a bayesian estimation, the aim being to find a pair (parameters, CTMC path) with likelihood as large as possible. approach. We refer the reader to Schnatter (2006) for a wider discussion on Markov switching models and the comparative advantages of the Bayesian approach. Standard priors are placed on the parameters space but, as the CTMC paths are unobserved, a large number of paths are drawn from a Dirichlet process placed as a prior on the path space of the CTMC. The complete model then appears as a Hierarchical Dirichlet Model (HDM), as in Ishwaran, James and Sun (2000) and Ishwaran and James (2002).

II. Main result

In this section we suppose the following assumption (**H**):

1. The two adapted processes $-c$ and d are local $\mathsf{M}(S)$-super martingales.
2. There exists some R^d -valued adapted process L such that for all $\mathsf{Q} \in \mathsf{Q}$, there exists some $\mathsf{R}^3 \otimes \mathsf{R}^d$ -valued predictable process ψ^{Q} with $dV_t^{\mathsf{Q}} = \psi_t^{\mathsf{Q}}.dL_t$.

Now we state the main result:
Under the assumption (**H**), for any positive Q -super martingale $(Y_t)_{t\in\mathsf{T}}$, there exists some $\mathsf{Q} \in \mathsf{Q}$, two predictable processes α^1, α^2 and an increasing process $(C_t)_{t\in\mathsf{T}}$ such that for all $t \in \mathsf{T}$ we have:

$$Y_t = Y_0 + \int_0^t \alpha_s^1 dS_s + \int_0^t \alpha_s^2 dX_s^{\mathsf{Q}} - C_t,$$

with

$$\mathsf{Q}'_t\left[\int_0^T \alpha_s^2 dX_s^{\mathsf{Q}}\right] \leq \int_0^t \alpha_s^2 dX_s^{\mathsf{Q}},$$

for all $\mathsf{Q}' \in \mathsf{Q}$.

Proof. To prove this theorem we should apply theorem 3.2 in [1] and state that:

1. The pricing set Q is m-stable.
2. The set $D = \{V^{\mathsf{Q}}, \mathsf{Q} \in \mathsf{Q}\}$ is progressively arbitrage free.
3. The pricing set Q is optionally m-stable wrt the finite viable portfolio V .
4. The set $int_A(D) := \{X \in int(D) : |X| \leq A\}$ is closed in L^0 for all random variable A with

$$int(D) = \left\{\int_0^T \beta_s^{\mathsf{Q}}.dV_s^{\mathsf{Q}} : \mathsf{Q} \in \mathsf{Q} \textit{ and } \beta^{\mathsf{Q}} \textit{ is an } \mathsf{R}^3\textit{-valued predictable pr}\right.$$

For assertion 1, let τ be a stopping time and $\mathsf{Q}^1, \mathsf{Q}^2 \in \mathsf{Q}$ with their respective densities Z^1, Z^2 . We define the probability Q with density $Z = Z_\tau^1 Z^2 / Z_\tau^2$. We shall prove that $\mathsf{Q} \in \mathsf{Q}$. Indeed the set $\mathsf{M}(S)$ is m-stable so $\mathsf{Q} \in \mathsf{M}(S)$ and for fixed $t \in \mathsf{T}$ we get for $t < \tau$,

$$\mathsf{Q}_t(X_T) = \mathsf{Q}_t^1\left[\mathsf{Q}_\tau^2(X_T)\right] \in [\mathsf{Q}_t^1(c_\tau), \mathsf{Q}_t^1(d_\tau)] \subseteq [c_t, d_t],$$

and for $t \geq \tau$,

$$\mathsf{Q}_t(X_T) = \mathsf{Q}_t^2(X_T) \in [c_t, d_t].$$

So $\mathsf{Q} \in \mathsf{Q}$.
For assertion 2, and thanks to assumption (**H2**) we get that:

$$D = \left\{\left(\int_0^t \psi_s^{\mathsf{Q}}.dL_s\right)_{t\in\mathsf{T}} : \mathsf{Q} \in \mathsf{Q}\right\}.$$

Then D is progressively arbitrage free. For assertion 3, we have

$$\mathsf{Q} = \cap_{t\in\mathsf{T}}\left\{\mathsf{Q} \prec \mathsf{P} : \mathsf{Q}_t(V) \in \{1\} \times \{S_t\} \times [c_t, d_t]\, a.s\right\}.$$

So by applying assertion 2 and corollary 2.8 in [1], we deduce that Q is optionally m-stable wrt the finite viable portfolio V . For assertion 4, and thanks to assumption (**H2**) we get that:

$$int(D) = \left\{\int_0^T \beta_s^{\mathsf{Q}} \psi_s^{\mathsf{Q}}.dL_s : \mathsf{Q} \in \mathsf{Q} \textit{ and } \beta^{\mathsf{Q}} \textit{ is predictable}\right\}$$

From [2], we deduce that the set $int(D)$ is closed in L^0 and so the set $int_A(D)$ is closed in L^0 .

Now for super hedging an European claim, we obtain the following result as an immediate consequence of theorem 2(`)@.

Corollary 1 *Under the assumption (* **H**), *for any positive random variable* Y *, there exists some*

$\mathsf{Q} \in \mathsf{Q}$, *two predictable processes* α^1, α^2 *and a positive random variable* C_T *such that:*

$$Y = \rho(Y) + \int_0^T \alpha_s^1 dS_s + \int_0^T \alpha_s^2 dX_s^{\mathsf{Q}} - C_T,$$

with $\rho(Y) = \sup_{\mathsf{Q}\in\mathsf{Q}} \mathsf{Q}(Y)$ and

$$\mathsf{Q}'_t\left[\int_0^T \alpha_s^2 dX_s^{\mathsf{Q}}\right] \le \int_0^t \alpha_s^2 dX_s^{\mathsf{Q}},$$

for all $\mathsf{Q}' \in \mathsf{Q}$.

III. Application

In this section we suppose that the process S is solution of the stochastic differential equation:

$$dS_t = \mu S_t dt + \sqrt{Y_t} S_t dW_t,$$

where the process Y satisfies the stochastic differential equation:

$$dY_t = a(b - Y_t)dt + \sigma\sqrt{Y_t} dB_t,$$

where W, B are two correlated Brownian motions, μ, a, b and σ are real parameters. Following notations of the last section we define $X_T = \max(S_T - K, 0)$ the payoff of an European call option with strike K, $c_t = -m_t(-X_T)$ and $d_t = \gamma m_t(X_T)$ with $0 < \gamma \le 1$. It is very known that the set $\mathsf{M}(S)$ is the set of probabilities Q^ψ with density given by

$$Z^\psi = \exp\left\{\int_0^T\left(-\frac{\mu}{Y_s} dW_s + \psi_s dW'_s\right) - \frac{1}{2}\int_0^T\left(\frac{\mu^2}{Y_s^2} + \psi_s^2\right)ds\right\}.$$

We obtain the following result For any positive random variable Y, there exists some $\mathsf{Q} \in \mathsf{Q}$, two predictable processes α^1, α^2 and a positive random variable C_T such that:

$$Y = \rho(Y) + \int_0^T \alpha_s^1 dS_s + \int_0^T \alpha_s^2 dX_s^{\mathsf{Q}} - C_T,$$

with $\rho(Y) = \sup_{\mathsf{Q}\in\mathsf{Q}} \mathsf{Q}(Y)$ and

$$\mathsf{Q}'_t\left[\int_0^T \alpha_s^2 dX_s^{\mathsf{Q}}\right] \le \int_0^t \alpha_s^2 dX_s^{\mathsf{Q}},$$

for all $\mathsf{Q}' \in \mathsf{Q}$.

In this section we consider an application of the last section using hierarchical model us follows:

IV. Markov regime switching with Dirichlet prior

In this section, we take $\overline{\alpha} = H$, the distribution of a continuous time Markov chain on a finite set of states and we propose a new hierarchical model that is specified, as an example, in the setting of mathematical finance. Of course, this can be similarly used in many other cases. We consider the Black-Scholes SDE in random environment with a Dirichlet prior on the path space of the chain, the states of the chain representing the environment due to the market. We model the stock price using a geometric Brownian motion with drift and variance depending on the state of the market. The state of the market is modeled as a continuous time Markov chain with a Dirichlet prior. In what follows, the notation σ will be used to denote the variance rather than the standard deviations.

The following notations will be adopted:

1. n will denote the number of observed data and also the length of an observed path.
2. M will denote the number of states of the Markov chain.
3. The state space of the chain will be denoted by $S = \{i : 1 \le i \le M\}$.
4. N will denote the number of simulated paths.
5. m will denote the number of distinct states of a path.

• The stock price follows the following SDE:

$$\frac{dS_t}{S_t} = \beta(X_t)dt + \sqrt{\sigma(X_t)}dB_t, \qquad t \ge 0,$$

where B_t is a standard Brownian motion. By the Ito's formula, the process $Z_t = \log(S_t)$ satisfies the SDE,

$$dZ_t = \mu(X_t)dt + \sqrt{\sigma(X_t)}dB_t, \qquad t \ge 0,$$

where $\mu(X_t) = \beta(X_t) - \frac{1}{2}\sigma(X_t)$. The observed data is of the form $Z_0, Z_1, \ldots, Z_n$.

• The process (X_t) is assumed to be a continuous time Markov process taking values in the set $S = \{i : 1 \le i \le M\}$. The transition probabilities of this chain are denoted by p_{ij}, $i, j \in S$ and the transition rate matrix is $Q_0 = (q_{ij})_{i,j\in S}$ with

$$\lambda_i > 0, \quad q_{ij} = \lambda_i p_{ij} \quad if \quad i \ne j, \quad and \quad q_{ii} = -\sum_{j\ne i} q_{ij}, \quad i, j \in S.$$

Then, conditional on the path $\{X_s, 0 \le s \le n\}, Y_t = Z_t - Z_{t-1} = \log(S_t/S_{t-1})$ are i.i.d. $\mathsf{N}(\mu_{X_t}, \sigma_{X_t})$, $t = 1, 2, \ldots, n.$

• For each $i = 1,2,\ldots,M,$ the priors on $\mu_i = \mu(i)$ and $\sigma_i = \sigma(i)$ are specified by

$$\mu_i \overset{ind}{:} \mathrm{N}(\theta,\tau^{\mu}), \quad with \quad \theta : \mathrm{N}(0,A), \quad A > 0, \tag{1}$$

$$\sigma_i \overset{ind}{:} \Gamma(\nu_1,\nu_2). \tag{2}$$

• The Markov chain $\{X_t, t \ge 0\}$ has prior $\mathrm{D}(\alpha\, H),$ where H is a probability measure on the path space of cadlag functions $D([0,\infty),S).$ The initial distribution according to H is the uniform distribution $\pi_0 = (1/M,\ldots,1/M),$ and the transition rate matrix is Q with $p_{ij} = 1/(M-1)$ and $\lambda_i = \lambda > 0$. Thus the Markov chain under Q will spend an exponential time with mean $1/\lambda$ in any state i and then jump to state $j \ne i$ with probability $1/(M-1).$

A realization of the Markov chain from the above prior is generated as follows: Generate a large number of paths $X_i = \{x_s^i : 0 \le s \le n\}$, $i = 1,2,\ldots,N,$ from $H.$ Generate the vector of probabilities $(p_i, i = 1,\ldots,N)$ from a Poisson Dirichlet distribution with parameter $\alpha,$ using stick breaking. Then draw a realization of the Markov chain from

$$p = \sum_{i=1}^{N} p_i \delta_{X_i}, \tag{3}$$

which is a probability measure on the path space $D([0,n),S).$ The parameter λ is chosen to be small so that the variance is large and hence we obtain a large variety of paths to sample from at a later stage. The prior for α is given by,

$$\alpha : \Gamma(\eta_1,\eta_2). \tag{4}$$

V. Estimation

Estimation is done using the simulation of a large number of paths of the Markov chain which will be selected according to a probability vector (generated by stick-breaking) and then using the blocked Gibbs sampling technique. This technique uses the posterior distribution of the various parameters.

To carry out this procedure we need to compute the following conditional distributions. We denote by $\mu,$ and σ , the current values of the vectors $(\mu_1,\mu_2,\ldots,\mu_n)$, $(\sigma_1,\sigma_2,\ldots,\sigma_n)$, respectively. Let Y be the vector of observed data $(Y_1,\ldots,Y_n).$ Let $X = (x_1,x_2,\ldots,x_n)$ be the vector of current values of the states of the Markov chain at times $t = 1,2,\ldots,n,$ respectively. Let $X^* = (x_1^*,\ldots,x_m^*)$ be the distinct values in $X.$

• **Conditional for** $\mu.$ For each $j \in X^*$ draw

$$\left(\mu_j \mid \sigma,X,\theta,Y\right) \overset{ind}{:} \mathrm{N}(\mu_j^*,\sigma_j^*), \tag{5}$$

where

$$\mu_j^* = \sigma_j^* \left(\sum_{t:X_t=j} \frac{Y_t}{\sigma_j} + \frac{\theta}{\tau^{\mu}} \right),$$

$$\sigma_j^* = \left(\frac{n_j}{\sigma_j} + \frac{1}{\tau^{\mu}} \right)^{-1},$$

and n_j being the number of times j occurs in $X.$ For each $j \in X \setminus X^*,$ independently simulate $\mu_j : \mathrm{N}(\theta,\tau^{\mu}).$

• **Conditional for** $\sigma.$ For each $j \in X^*$ draw

$$\left(\sigma_j \mid \mu,K,Y\right) \overset{ind}{:} \Gamma(\nu_1 + \frac{n_j}{2}, \nu_{2,j}^*), \tag{6}$$

where

$$\nu_{2,j}^* = \nu_{2,j} + \sum_{t:X_t=j} \frac{(Y_t-\mu_j)^2}{2}.$$

Also for each $j \in X \setminus X^*,$ independently simulate $\sigma_j : \Gamma(\nu_1,\nu_2).$

• **Conditional for** $X.$

$$\left(X \mid p\right) : \sum_{i=1}^{N} p_i^* \delta_{X_i}, \tag{7}$$

where

$$p_i^* \propto \prod_{g=1}^{m} \Big(\prod_{\{d,x_d^{i,*}=g\}} \frac{1}{(2\sigma_g\pi)^{1/2}} e^{-\frac{1}{2\sigma_g}(Y_d-\mu_g)^2} \Big) p_i, \tag{8}$$

where $(x_1^{i,*},\ldots,x_m^{i,*})$ denote the current $m = m(i)$ unique values of $X_i,$ $i = 1,\ldots,N.$

• **Conditional for** $p.$

$$p_1 = V_1^*, \; and \; p_k = (1-V_1^*)\cdots(1-V_{k-1}^*)V_k^*, \; k = 2,3,\ldots,N-1, \tag{9}$$

where

$$V_k^* \overset{ind}{:} \beta(1+r_k, \alpha + \sum_{\ell=k+1}^{N} r_\ell),$$

r_k being the number of x_l^i 's which equal k .

- **Conditional for α .**

$$(\alpha \mid p) : \Gamma\left(N + \eta_1 - 1, \eta_2 - \sum_{i=1}^{N-1} \log(1 - V_i^*) \right),$$

where the V^* values are those obtained in the simulation of p in the above step.

- **Conditional for θ.**

$$(\theta \mid \mu) : \mathrm{N}(\theta^*, \tau^*), \qquad (10)$$

where

$$\theta^* = \frac{\tau^*}{\tau^\mu} \sum_{j=1}^{M} \mu_j ,$$

and

$$\tau^* = \left(\frac{M}{\tau^\mu} + \frac{1}{A} \right)^{-1} .$$

5.1 Real data

We have applied our algorithm to the index data of the Europian Stock Exchange. For this dataset we have, $n = 250$, $\Delta t = 1$, and we deal with $N = 100$ of paths while Gamma(2, 4) is the prior for α.

With the above choice, we obtain 6 regimes for which the estimates for the mean, variance and stationary probabilities are as follows:

	R 1	R 2	R 3	R 4	R 5	R 6
μ	0.001124	-0.009479	0.000629	-0.004579	0.000829	0.001109
σ	2.9132 e-05	7.2166 e-05	2.3023 e-05	7.3800 e-05	1.186 e-05	3.3372 e-05
π	20 %	3 %	29 %	5 %	10 %	33 %

The most frequent Markov chain path, its parameters λ_i s and the matrix of transition probability $(p_{i,j})_{1 \le i \ne j \le 6}$ are respectively equal to:

3 5 3 6 3 6 3 6 1 6 5 1 3 6 3 5 3 3 6 6 5 6 3 6 1 1 4 1 6 1 3 3 6 6 6 3 1 3 3 3 6 3 3 3 4 5 6 6 6 6 4 6 1 1 1 6 6 6 6 6 1 3 3 3 1 6 1 3 3 5 6 3 3 1 6 5 4 1 3 6 4 6 3 3 5 6 3 6 2 3 6 1 3 3 6 1 6 6 5 5 1 1 5 3 5 3 3 6 1 6 5 6 1 6 6 3 1 6 3 1 1 6 2 3 6 6 6 3 3 2 6 6 6 1 3 3 6 6 3 1 3 6 6 1 6 6 1 1 6 1 5 3 5 1 3 5 3 4 1 3 3 5 3 1 3 6 6 6 1 3 5 6 5 3 3 6 3 6 1 3 5 6 6 6 5 1 6 3 3 1 1 6 6 6 3 6 1 3 6 3 6 6 6 6 6 3 6 3 6 6 4 6 3 6 1 1 6 4 6 1 3 4 3 6.

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0.8	1	0.7	1	0.95	0.75

$$\begin{pmatrix} & 0 & 0.48 & 0.03 & 0.06 & 0.42 \\ 0 & & 0.66 & 0 & 0 & 0.33 \\ 0.16 & 0.02 & & 0.062 & 0.2 & 0.54 \\ 0.375 & 0 & 0 & & 0.125 & 0.5 \\ 0.157 & 0 & 0.42 & 0.052 & & 0.36 \\ 0.36 & 0.038 & 0.384 & 0.077 & 0.134 & \\ & & & & & \end{pmatrix}$$

VI. Conclusion

A Bayesian approach to estimation for a regime switching geometric Brownian motion is proposed. The algorithm while being computationally intensive is able to segregate the different regimes based on the drift and volatility, thus giving useful insights into the behavior of the market. It has been observed empiricaly that markets fluctuate between periods of high, moderate and low volatilities. The above estimation procedure provides a clear quantitative picture of the number of regimes and an estimate of the drifts and volatilities in these regimes. Estimation of current market state is also easier using the algorithm proposed compared to models using continuous stochastic volatility models. Given an estimate of the regime, the algorithm also gives an idea of likely duration for which the regime is likely to persist and the distribution of the regimes that may follow.

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